N92-22064

THE ORIGIN OF NON-CLASSICAL EFFECTS IN A ONE-DIMENSIONAL SUPERPOSITION OF COHERENT STATES

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Abstract

We investigate the nature of the quantum fluctuations in a light field created by the superposition of coherent fields. We give a physical explanation (in terms of Wigner functions and phase-space interference) why the one-dimensional superposition of coherent states in the direction of the x-quadrature leads to the squeezing of fluctuations in the y-direction, and show that such a superposition can generate the squeezed vacuum and squeezed coherent states.

1 Introduction

The coherent states are always associated with the "most" classical states one can imagine in the framework of quantum theory [1]. In the present Lecture we will study the quantum interference between coherent states and how such interference leads to generation of states whose properties are as far as one can imagine from "classical" states. In particular a one-dimensional superposition of coherent states can exhibit sub-Poissonian photon statistics or squeezing. In our Lecture we will concentrate mainly on squeezing which appears as a consequence of quantum interference between coherent states.

Light squeezing (for recent reviews see [2] as well as topical issues of *JOSA B* [3] and *J. Mod. Opt.* [4]) remains a central topic in quantum optics. Generation of squeezed light has been reported by various groups [5-11] and offers new opportunities for the utilization of light with reduced quadrature noise in interferometry, fiber optics communications and high-precision experiments. Most studies of squeezed states have concentrated on those states generated by quadratic field interaction (e.g. parametric amplification). Recently it has been shown by Wódkiewicz and coworkers [12] that a superposition of two number states (for instance, the vacuum state and the one- or two-photon states) of a single mode electromagnetic field exhibits interesting non-classical properties. In particular, squeezing of the variances of the quadrature operators can be seen (although not necessarily of the quadratic, minimum uncertainty state quality). A superposition of a

finite number of coherent states has also been studied [13-17]. In particular, Hillery [13] has studied the superposition of two coherent states $|\alpha\rangle + |-\alpha\rangle$ (the so-called "even coherent state" [14]) in connection with amplitude-squared squeezing. Yurke and Stoler [15] have shown that such a superposition of coherent states can arise as a consequence of propagation of coherent light through an amplitude-dispersive medium. It has been shown that the even coherent states exhibit ordinary (second order) squeezing as well as fourth order squeezing [16]. In a recent paper, Janszky and Vinogradov [17] extended the idea of superposition of coherent states and investigated the quadrature variances of a continuous one-dimensional superposition of coherent states. They have shown that such a superposition of coherent states can lead to significant reduction of fluctuations in one of the quadratures.

At first sight, the result of Janszky and Vinogradov seems quite remarkable, when reinterpreted in terms of interference in phase space: a superposition of coherent states in the direction of the x-quadrature leads to a suppression in the fluctuations in the y-direction, whereas naively one would expect that the quantum interference relevant to this superposition would modify the fluctuations in the original x-direction.

In the present Lecture we give a physical explanation (in terms of the Wigner function and a phase-space formalism [18-20]) of the origin of this noise suppression and squeezing for a one-dimensional superposition of coherent states. We further demonstrate that a suitable Gaussian superposition of coherent states not only can be squeezed, but is actually a representation of the minimum-uncertainty squeezed vacuum state.

2 Simple example

We start our Lecture with a simple example considering a superposition of two coherent states $|\alpha_1\rangle$ and $|\alpha_2\rangle$

$$|\Psi\rangle = A^{1/2}\{|\alpha_1\rangle + |\alpha_2\rangle\}, \tag{1}$$

where A is a normalization constant

$$A^{-1} = 2(1 + \mathrm{Re}\langle\alpha_1|\alpha_2\rangle).$$

The coherent state $|\alpha\rangle$ can be obtained by shifting the vacuum state $|0\rangle$ by the displacement operator $\hat{D}(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$:

$$|\alpha\rangle = \hat{D}(\alpha)|0\rangle,$$

where $\hat{a}^\dagger$ ($\hat{a}$) is the creation (annihilation) operator of a photon.

The density matrix corresponding to the superposition of coherent states (1) is given by the following expression

$$\hat{\rho} = A(|\alpha_1\rangle\langle\alpha_1| + |\alpha_2\rangle\langle\alpha_2| + |\alpha_1\rangle\langle\alpha_2| + |\alpha_2\rangle\langle\alpha_1|), \tag{2}$$

while the density matrix describing statistical mixture of two coherent states $|\alpha_1\rangle$ and $|\alpha_2\rangle$ is

$$\hat{\rho}_M = p_1|\alpha_1\rangle\langle\alpha_1| + p_2|\alpha_2\rangle\langle\alpha_2| \tag{3}$$

where p_i is the probability to find the system in the state $|\alpha_i\rangle$. These probabilities are normalized to 1.

2.1 Wigner functions

Now we introduce the notion of the Wigner function through the characteristic function $C^{(W)}(\xi)$, which is

associated with the symmetrical order of the bosonic (photon) operators and is given by the relation [21]

$$C^{(W)}(\xi) = \mathrm{Tr}\left[\hat{\rho}\exp(i\xi\hat{a}^{\dagger} + i\xi^{*}\hat{a})\right]. \qquad (4)$$

The Wigner function is defined as the Fourier transform of the characteristic function $C^{(W)}(\xi)$:

$$W(\beta) = \pi^{-2}\int d^{2}\xi \exp[-i(\xi\beta^{*} + \xi^{*}\beta)]C^{(W)}(\xi). \qquad (5)$$

The Wigner function corresponding to the superposition of two coherent states (1) can be written in the form

$$W(\beta) = A\,(W_1 + W_2 + W_{12}) \qquad (6.a)$$

where

$$W_i = \frac{2}{\pi}\exp(-2|\alpha_i - \beta|^2); \qquad (6.b)$$

and

$$\begin{aligned} W_{12} = \frac{2}{\pi}\exp\left[-\frac{1}{2}\left(|\alpha_1|^2 + |\alpha_2|^2\right)\right] \\ \times\{\exp\left[\alpha_2\alpha_1^{*} - 2(\beta - \alpha_2)(\beta^{*} - \alpha_1^{*})\right] \qquad (6.c) \\ + \exp\left[\alpha_1\alpha_2^{*} - 2(\beta - \alpha_1)(\beta^{*} - \alpha_2^{*})\right]\} \end{aligned}$$

The terms W_i are the Wigner functions corresponding to the coherent states $|\alpha_i\rangle$, while the term W_{12} arises due to the quantum interference between coherent states under consideration.

The Wigner function for the statistical mixture (3) is given by the relation

$$W_M = p_1 W_1 + p_2 W_2 \qquad (7)$$

and it does not contain the term describing the quantum interfence between coherent states.

2.2 Even coherent states

To simplify our task we will suppose that $\alpha_1 = -\alpha_2 = \alpha$, where α is a real parameter. In this case we obtain from (1) the following state

$$|\Psi\rangle = A^{1/2}\{|\alpha\rangle + |-\alpha\rangle\}, \qquad (8)$$

with the normalization constant

$$A^{-1} = 2[1 + \exp(-2\alpha^2)].$$

The state (8) is called [13,14] the even coherent state. The Wigner function corresponding to this state can be found using the general expression (6) and is presented in Figure 1, where $x = \mathrm{Re}\beta$ and $y = \mathrm{Im}\beta$. If we compare this function with the Wigner function (see Figure 2) corresponding to the statistical mixture of states $|\alpha\rangle$ and $|-\alpha\rangle$ described by the density matrix

$$\hat{\rho}_M = \frac{1}{2}\left(|\alpha\rangle\langle\alpha| + |-\alpha\rangle\langle-\alpha|\right) \qquad (9)$$

we can directly observe that the term W_{12} corresponding to the quantum interference between states $|\alpha\rangle$ and $|-\alpha\rangle$ should play an important rôle in statistical properties of superpositions of coherent states.

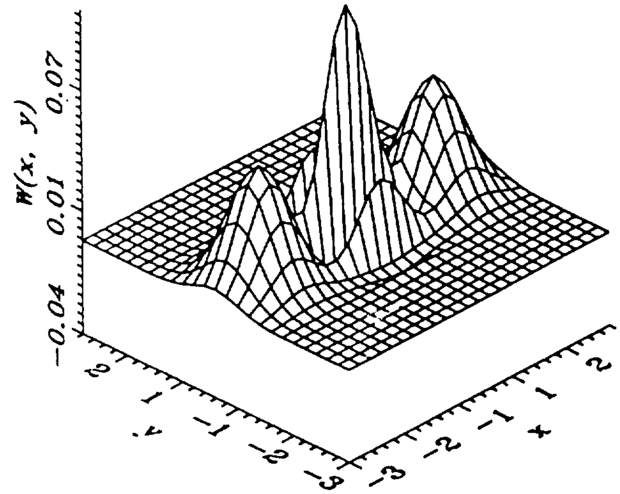

Figure 1: Wigner function corresponding to the even coherent state (8) with $\alpha = 2$. The rôle of the interference term is transparent.

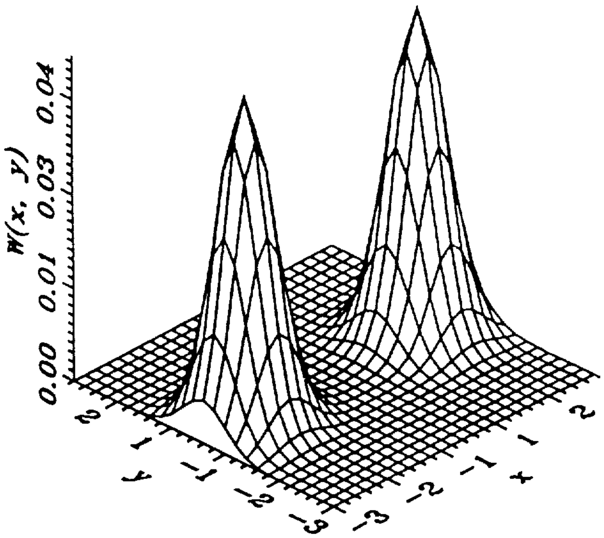

Figure 2: Wigner function corresponding to the statistical mixture (9) with $\alpha = 2$ and $p_1 = p_2 = 1/2$

2.3 Quadrature squeezing

The quadrature operators $\hat{a}_1$ and $\hat{a}_2$ corresponding to the creation and annihilation operators $\hat{a}^\dagger$ and $\hat{a}$ are defined as:

$$\hat{a}_1 = \frac{\hat{a} + \hat{a}^\dagger}{2} \quad ; \quad \hat{a}_2 = \frac{\hat{a} - \hat{a}^\dagger}{2i}. \tag{10}$$

We can easily find that variances of these operators

$$\langle (\Delta \hat{a}_i)^2 \rangle = \langle \hat{a}_i^2 \rangle - \langle \hat{a}_i \rangle^2, \tag{11}$$

in the statistical mixture (9) are:

$$\langle (\Delta \hat{a}_1)^2 \rangle_M = \frac{1}{4} + \alpha^2,$$

and

$$\langle (\Delta \hat{a}_2)^2 \rangle_M = \frac{1}{4}.$$

From above it follows that the fluctuations in the $\hat{a}_1$ quadrature are larger in the case of the statistical mixture compared to the vacuum-state (or the coherent-state) value, which is equal to 1/4. Fluctuations in $\hat{a}_2$ remain the same for both the statistical mixture (9) as well as for the coherent state $|\alpha\rangle$ (or $|-\alpha\rangle$).

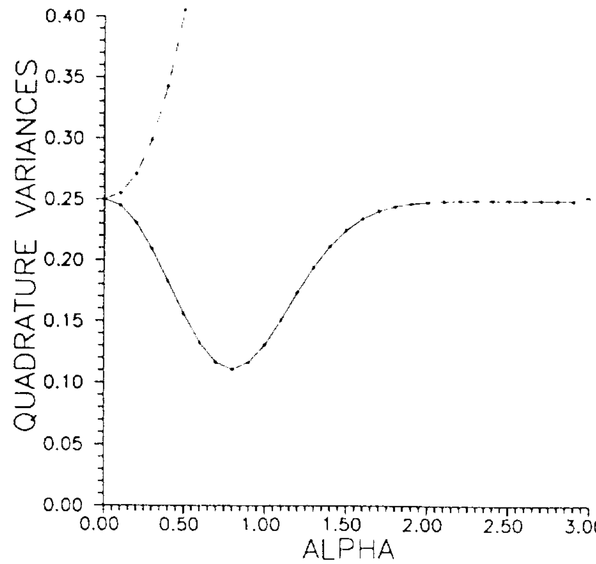

Figure 3: Quadrature variances given by equations (12) versus parameter α. The dashed line corresponds to $\langle (\Delta \hat{a}_1)^2 \rangle$ and the solid line corresponds to $\langle (\Delta \hat{a}_2)^2 \rangle$.

On the other hand, for the even coherent state (8) we find the reduction of fluctuations in $\hat{a}_2$ quadrature (i.e. in the y-direction in the phase space – see Figure 1):

$$\langle (\Delta \hat{a}_2)^2 \rangle = \frac{1}{4} - \frac{\alpha^2 \exp(-2\alpha^2)}{1 + \exp(-2\alpha^2)}. \tag{12.a}$$

Simultaneously fluctuations in $\hat{a}_1$ are enhanced:

$$\langle (\Delta \hat{a}_1)^2 \rangle = \frac{1}{4} + \frac{\alpha^2}{1 + \exp(-2\alpha^2)}. \tag{12.b}$$

Variances $\langle (\Delta \hat{a}_i)^2 \rangle$ versus the parameter α (which is related to the intensity of the even coherent state) are plotted in Figure 3. We see that the maximum reduction in the fluctuations can be obtained for quite small values of α. Reduction of fluctions below the vacuum-state (or coherent-state) level is called quadrature squeezing. From the above it follows that quadrature squeezing can emerge as a consequence of the quantum interference between coherent states. We should note that even coherent states (8) exhibit not only quadrature squeezing, but also higher-order squeezing as well as amplitude squared squeezing [13,16].

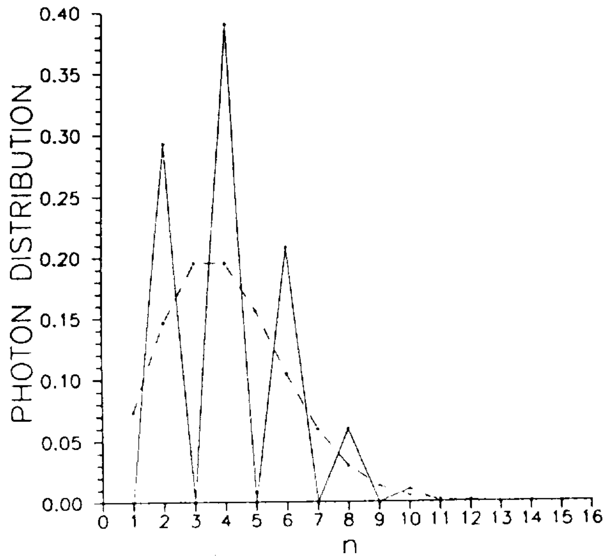

Figure 4: Photon number distribution for a superposition of two coherent states (8) with amplitude $\alpha = 2$. The dashed line is the distribution of the corresponding statistical mixture (9).

2.4 Photon number distribution

Here we discuss briefly properties of the photon number distribution of the statistical mixture (3) as well as the superposition (2) of two coherent states. The photon number distribution is defined as

$$P_n = \langle n|\hat{\rho}|n\rangle. \tag{13}$$

and can be evaluated easily for both the statistical mixture (3)

$$P_n^M = \frac{1}{n!}\left\{p_1|\alpha_1|^{2n}e^{-|\alpha_1|^2} + p_2|\alpha_2|^{2n}e^{-|\alpha_2|^2}\right\} \tag{14}$$

and for the superposition of coherent states (2):

$$P_n = \frac{A}{n!}\left\{|\alpha_1|^{2n}e^{-|\alpha_1|^2} + |\alpha_2|^{2n}e^{-|\alpha_2|^2}\right.$$

$$\left.+[(\alpha_1\alpha_2^*)^n + (\alpha_2\alpha_1^*)^n]\exp[-\frac{1}{2}(|\alpha_1|^2 + |\alpha_2|^2)]\right\} \tag{15}$$

In the case of the statistical mixture the photon number distribution (14) is just the sum of two Poissonian distributions corresponding to two independent coherent states $|\alpha_1\rangle$ and $|\alpha_2\rangle$. On the other hand, in the case of a superposition of coherent states $|\alpha_1\rangle$ and $|\alpha_2\rangle$ the term corresponding to the quantum interference plays an important rôle. To see this clearly we will assume that $\alpha_1 = -\alpha_2 = \alpha$. In this case the statistical mixture (9) has just the Poissonian photon number distribution

$$P_n^M = \frac{|\alpha|^{2n}}{n!}e^{-|\alpha|^2}. \tag{16}$$

The superposition of coherent states under consideration (i.e. the even coherent state) has the following photon number distribution:

$$P_n = \begin{cases} \frac{2\exp(-|\alpha|^2)}{1+\exp(-|\alpha|^2)}\frac{|\alpha|^{2n}}{n!} & \text{if } n = 2m \\ 0 & \text{if n=2m+1,} \end{cases} \tag{17}$$

The oscillations in P_n are very similar to those in the case of the squeezed vacuum discussed by Schleich and Wheeler [18]. Generally, these oscillations are due to quantum interference in the phase space.

3 Continuous superposition of CS

Now we will discuss the properties of one-dimensional continuous superposition of coherent states. Recently Janszky and Vinogradov [17] defined the continuous superposition $|\xi\rangle$ of coherent states $|\alpha\rangle$ in the following way

$$|\xi\rangle \equiv C_F\int_{-\infty}^{\infty} F(\alpha,\xi)|\alpha\rangle d\alpha, \tag{18}$$

where the coherent state amplitude α is supposed to be real. The normalization constant C_F is defined as:

$$C_F^{-2} = \int\int_{-\infty}^{\infty} F(\alpha,\xi)F(\alpha',\xi)\exp[-(\alpha-\alpha')^2/2]d\alpha d\alpha'. \tag{19}$$

With the superposition state $|\xi\rangle$ given by equation (18) one can find expressions for the mean values of products of the creation $\hat{a}^\dagger$ and the annihilation $\hat{a}$ operators of the field mode in the following form:

$$\langle(\hat{a}^\dagger)^m \hat{a}^n\rangle = \int\int_{-\infty}^{\infty} F(\alpha,\xi)F(\alpha',\xi)$$

$$\times \exp[-(\alpha-\alpha')^2/2]\alpha^m(\alpha')^n d\alpha d\alpha'. \qquad (20)$$

In particular, if $F(\alpha,\xi)$ is taken to be the Gaussian function

$$F(\alpha,\xi) = \exp\left[-\frac{(1-\xi)}{2\xi}\alpha^2\right] \qquad (21)$$

with $\xi \in (0,1)$ and

$$C_F^{-1} = \pi^{-1/2}\frac{(1-\xi^2)^{1/4}}{(2\xi)^{1/2}}, \qquad (22)$$

then one can find for the variances of the quadrature operators $\hat{a}_1$, $\hat{a}_2$ given by eq. (10) the following expressions

$$\langle(\Delta\hat{a}_1)^2\rangle = \left(\frac{1}{4}\right)\frac{(1+\xi)}{(1-\xi)}; \qquad (23.a)$$

$$\langle(\Delta\hat{a}_2)^2\rangle = \left(\frac{1}{4}\right)\frac{(1-\xi)}{(1+\xi)}. \qquad (23.b)$$

From this one can conclude that the states $|\xi\rangle$: **i)** belong to the class of the minimum uncertainty states; **ii)** the fluctuations in the "second" quadrature are reduced below the shot noise limit.

We see, therefore, that the one-dimensional superposition (with Gaussian distribution) of coherent states leads to states exhibiting a large degree of squeezing (in the limit $\xi \to 1$). We now demonstrate by direct calculations that if $F(\alpha,\xi)$ is the Gaussian function (21), then the state $|\xi\rangle$ is equal to the squeezed vacuum state generated by the action of the squeeze operator $\hat{S}(\hat{a}^\dagger,\hat{a},\xi)$:

$$\hat{S}(\hat{a}^\dagger,\hat{a},\xi) = \exp\left[\frac{r}{2}(\hat{a}^\dagger)^2 - \frac{r}{2}\hat{a}^2\right],$$

$$\xi = \tanh r, \qquad (24)$$

on the vacuum state $|0\rangle$ of the field mode. To do so we decompose the coherent state $|\alpha\rangle$ in equation (18) into number states:

$$|\alpha\rangle = \exp(-\alpha^2/2)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{(n!)}}|n\rangle,$$

and exchange the order of integration and summation procedures, i.e. we rewrite equation (18) in the form

$$|\xi\rangle = C_F\sum_{n=0}^{\infty}\frac{1}{\sqrt{(n!)}}|n\rangle$$

$$\times\int_{-\infty}^{\infty} d\alpha\,\alpha^n \exp\left[-\frac{1}{2\xi}\alpha^2\right]. \qquad (25)$$

After performing the integration on the r.h.s of equation (25) we find

$$|\xi\rangle = (1-\xi^2)^{1/4}\sum_{n=0}^{\infty}\frac{[(2n)!]^{1/2}}{2^n n!}\xi^n|2n\rangle, \qquad (26)$$

from which it follows that the one-dimensional superposition of coherent states (18) with the distribution function (21) is *identical* to the squeezed vacuum state:

$$|\xi\rangle = C_F\int_{-\infty}^{\infty} F(\alpha,\xi)d\alpha\hat{D}(\alpha)|0\rangle = \hat{S}(\xi)|0\rangle. \qquad (27)$$

We should stress here that the last equation describes the relation between the states, but not between the displacement and the squeeze operators themselves.

3.1 Origin of squeezing

We next provide a physical explanation of the origin of the squeezing generated by such a superposition. We address the question of how the one-dimensional superposition of coherent states in direction of the x-quadrature (corresponding to the operator $\hat{a}_1$) leads to squeezing of the fluctuations in y-direction (associated with the quadrature operator $\hat{a}_2$). To do so we use the Wigner-function phase-space formalism: we define the Wigner function $W(\beta)$ in the following way. First,

we introduce a "generalized" characteristic function $\tilde{C}^{(W)}(\alpha, \alpha', \zeta)$:

$$\tilde{C}^{(W)}(\alpha, \alpha', \zeta) \equiv \langle \alpha' | \hat{D}(\zeta) | \alpha \rangle$$

$$= \exp\left[-\frac{1}{2}|\zeta|^2 + i\zeta\alpha' + i\zeta^*\alpha - \frac{1}{2}(\alpha - \alpha')^2\right], \quad (28)$$

and the "generalized" Wigner function $\tilde{W}(\alpha, \alpha', \beta)$:

$$\tilde{W}(\alpha, \alpha', \beta) = \pi^{-2} \int d^2\zeta$$

$$\times \exp[-i(\zeta^*\beta + \zeta\beta^*)]\tilde{C}^{(W)}(\alpha, \alpha', \zeta) \quad (29)$$

$$= \frac{2}{\pi} \exp\left[\frac{1}{2}(\beta^* - \beta - \alpha' + \alpha)^2\right]$$

$$\times \exp\left[-\frac{1}{2}(\beta + \beta^* - \alpha - \alpha')^2 - \frac{1}{2}(\alpha - \alpha')^2\right].$$

The Wigner function $W(\beta)$ can now be expressed in a very simple form:

$$W(\beta) = C_F^2 \int\int_{-\infty}^{\infty} d\alpha d\alpha' F(\alpha) F(\alpha') \tilde{W}(\alpha, \alpha', \beta). \quad (30)$$

To make our discussion more transparent we will first analyze in detail the simple superposition of two coherent states $|\alpha\rangle$ and $|-\alpha\rangle$ and the vacuum state $|0\rangle$, i.e. we will study the state [17]

$$|\xi\rangle = C_F \{|\alpha\rangle + p|0\rangle + |-\alpha\rangle\}, \quad (31)$$

which can be obtained from equation (18) with a weight function $F(x) = \delta(x - \alpha) + p\delta(x) + \delta(x + \alpha)$. The normalization constant in this case is given by the relation:

$$C_F^{-2} = 2 + p^2 + 4p\exp(-\alpha^2/2) + 2\exp(-2\alpha^2). \quad (32)$$

One can easily find the variance of the quadrature operator $\hat{a}_2$ for the state (31):

$$\langle(\Delta\hat{a}_2)^2\rangle = \frac{1}{4}\{1 - 4C_F^2\alpha^2$$

$$\times[2\exp(-2\alpha^2) + p\exp(-\alpha^2)]\}, \quad (33)$$

from which it follows that a high degree of squeezing (up to 74%) can be obtained for the optimum case, $\alpha = 1.57$ and $p = 1.35$.

The Wigner function of the state (31) can be expressed as the sum of two terms:

$$W(\beta) = W_{cl}(\beta) + W_{quant}(\beta), \quad (34)$$

where

$$W_{cl}(\beta) = \frac{2C_F^2}{\pi}\{\exp[-2(x - \alpha)^2 - 2y^2]$$

$$+ \exp[-2(x + \alpha)^2 - 2y^2] + p^2\exp[-2x^2 - 2y^2]\} \quad (35)$$

and

$$W_{quant}(\beta) = \frac{2C_F^2}{\pi}\{2\cos(4\alpha y)\exp[-2x^2 - 2y^2]$$

$$+p\cos(2\alpha y)\left[\exp\left(-2(x - \alpha/2)^2 - 2y^2\right)\right.$$

$$\left.+ \exp\left(-2(x + \alpha/2)^2 - 2y^2\right)\right]\}. \quad (36)$$

The normalization constant C_F in this case is given by equation (32) and $x = \mathrm{Re}\beta$; $y = \mathrm{Im}\beta$. The function $W_{cl}(\beta)$ is equal (up to normalization factors) to the sum of the independent Wigner functions of the vacuum state and two coherent states and can be identified with the Wigner function of the statistical mixture of coherent states and the vacuum state described by the density matrix

$$\hat{\rho}_M = p_1|0\rangle\langle 0| + p_2|\alpha\rangle\langle\alpha| + p_3|-\alpha\rangle\langle-\alpha| \quad (37)$$

with properly chosen parameters p_i.

This function is plotted in Figure 5a, from which it is obvious that $W_{cl}(\beta)$ is positive for any value of x and y. The phase-space contour lines of this function are plotted in Figure 5b. In contrast to the function

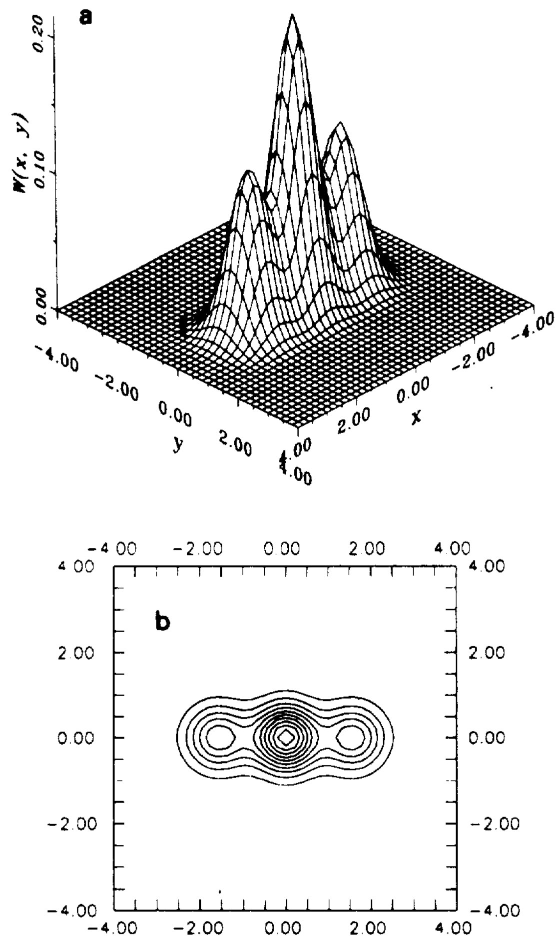

Figure 5: Function $W_{cl}(x, y)$ given by Equation (35) representing the part of the Wigner function of the superposition state (31) is plotted for $\alpha = 1.57$ and $p = 1.35$ (a). In Figure 5b the phase-space contours corresponding to this function are plotted.

$W_{cl}(\beta)$, the function $W_{quant}(\beta)$ can be negative. This function describes in phase-space the quantum interference effects between the states $|\alpha\rangle$, $|-\alpha\rangle$ and $|0\rangle$. The quantum interference is responsible the appearance of the cosine terms in the y-direction, and these oscillating terms are responsible for: **1)** negative values of the function $W_{quant}(\beta)$ (see Figure 6a) as well as the total Wigner function $W(\beta)$ (Figure 7a); **2)** squeezing of the variance of the quadrature operator in the y-direction, which is clearly seen in Figures 6b and 7b.

This simple example helps us to understand the nature of squeezing in the one-dimensional superposition of coherent states. The squeezing arises as a consequence of quantum interference between the macroscopically distinguishable states. Generally, if more states are involved in the superposition, a higher degree of squeezing (depending on the appropriate shape of the distribution $F(\alpha)$) can be obtained for the same mean value of photons in the mode.

Now we turn our attention to the displacement of the superposition state such that there is a mean field amplitude. We show that a one-dimensional superposition of coherent states with the distribution function $F(\alpha, \xi, \beta)$ centered at a non-zero value of α is equal to the squeezed coherent state. We take for our distribution function $F(\alpha, \xi, \beta)$ the displaced form

$$F(\alpha, \xi, \beta) = \exp\left[-\frac{(1-\xi)}{2\xi}(\alpha - x_0)^2\right], \tag{38}$$

with the normalization constant C_F given by equation (22) and with a displacement

$$x_0 = \left(\frac{1+\xi}{1-\xi}\right)^{1/2}\beta.$$

In this case from equation (18) we obtain for the state

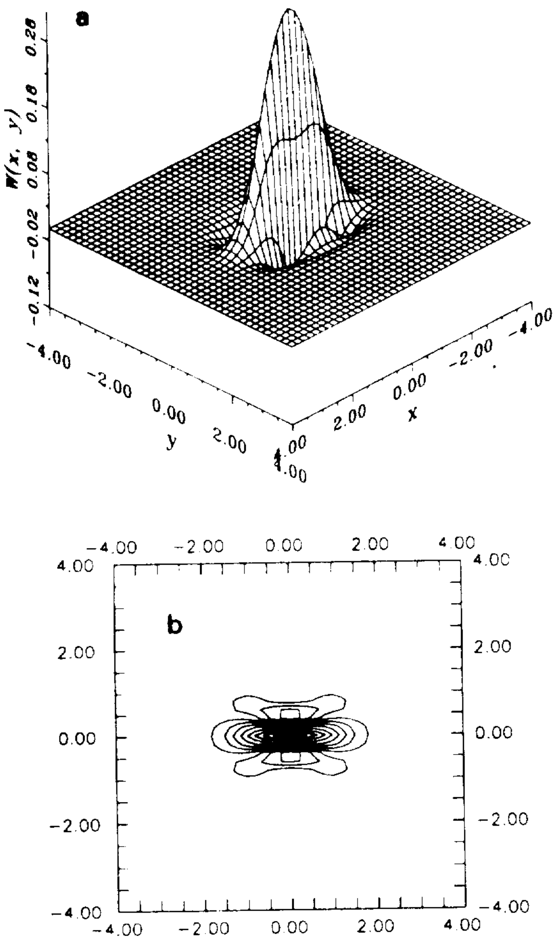

Figure 6: Function $W_{quant}(x, y)$ (36) representing the interference between states in phase space is plotted for $\alpha = 1.57$ and $p = 1.35$ (a). In Figure 6b the phase-space contours corresponding to this function are plotted.

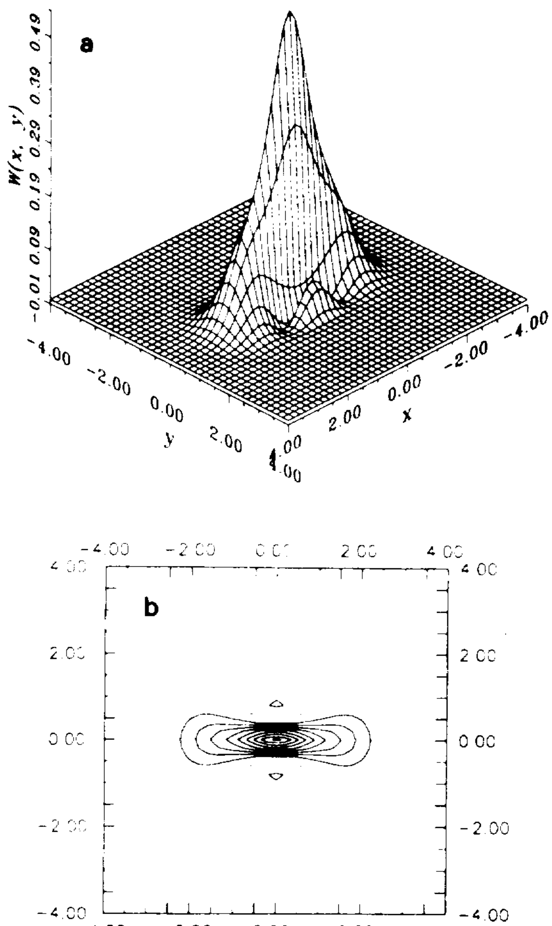

Figure 7: The total Wigner function $W(x, y)$ is plotted for $\alpha = 1.57$ and $p = 1.35$ (a). In Figure 7b the phase-space contours corresponding to this function are plotted.

$|\xi\rangle$ the following expression:

$$|\xi\rangle = (1-\xi^2)^{1/4} \exp\left[-\frac{(1-\xi)}{2}x_0^2\right] \times \sum_{n=0}^{\infty} \sqrt{(n!)}\,[(1-\xi)x_0]^n \times \left\{ \sum_{k=0}^{\|n/2\|} \frac{1}{(n-2k)!k!} \left(\frac{\xi}{2(1-\xi)^2 x_0^2}\right)^k \right\} |n\rangle, \quad (39)$$

where $\|x\|$ denotes the greatest integer less than or equal to x. Using the new parametrization:

$$\mu = \frac{1}{(1-\xi^2)^{1/2}} \quad ; \quad \nu = \frac{-\xi}{(1-\xi^2)^{1/2}}, \quad (40)$$

with $\mu^2 - \nu^2 = 1$ and $|\mu| > |\nu|$, we can rewrite equation (39) in the form:

$$|\xi\rangle = \mu^{-1/2} \exp\left[-\frac{(1-\nu/\mu)}{2}\beta^2\right] \sum_{n=0}^{\infty} \frac{1}{\sqrt{(n!)}} \left(\frac{\nu}{2\mu}\right)^{n/2} H_n(\beta/\sqrt{2\mu\nu})|n\rangle, \quad (41)$$

where $H_n(x)$ is the Hermite polynomial. It is obvious that the last expression obtained describes precisely the squeezed coherent state as defined by Yuen [22], i.e., we have explicitly proved that

$$\hat{S}(\xi)\hat{D}(\beta)|0\rangle = C_F \int_{-\infty}^{\infty} d\alpha\, F(\alpha,\xi,\beta)\hat{D}(\alpha)|0\rangle. \quad (42)$$

In other words, we can construct, through a one-dimensional superposition of coherent states with a properly chosen distribution function, the squeezed coherent state. Obviously, the physical reason for squeezing is the same as for the case of squeezed vacuum state discussed earlier. It is amusing that a superposition of the most classical of field states, the coherent states, can through the action of quantum interference, generate the archetypal nonclassical field states – the squeezed vacuum and the squeezed coherent state.

It can also be shown that the squeezed number state [23] defined as a result of a action of the squeezing operator $\hat{S}(\xi)$ on the number state $|n\rangle$ can be constructed as a one-dimensional superposition of displaced number states [24], the states obtained through the action of the displacement operator on the number state, that is

$$\hat{S}(\xi)|n\rangle = C_F \int_{-\infty}^{\infty} F(\alpha,\xi) d\alpha \hat{D}(\alpha)|n\rangle, \quad (43)$$

where function $F(\alpha,\xi)$ and the normalization constant C_F are given by the Equations (21) and (22), respectively.

4 Discussion

In our Lecture we discussed the rôle of the quantum interference in the origin of squeezing in the one-dimensional superposition of coherent states. With the aim to make the discussion as clear as possible we started our Lecture with a simple example of superposition of just two coherent states $|\alpha\rangle$ and $|-\alpha\rangle$. Finishing the Lecture we return to this simple example but we take into account the relative phase between the coherent states under consideration, i.e. we will study the following superposition

$$|\Psi\rangle = A^{1/2}\left\{|\alpha\rangle + e^{i\phi}|-\alpha\rangle\right\}, \quad (44)$$

with the normalization constant

$$A^{-1} = 2\left(1 + \cos\phi e^{-2\alpha^2}\right).$$

We will show that the phase ϕ plays a crucial rôle in the character of the quantum interference between coherent states.

First of all we write down the corresponding Wigner function for the state (44). This function can be expressed as a sum of two terms:

$$W(\beta) = W_{cl}(\beta) + W_{quant}(\beta), \quad (45)$$

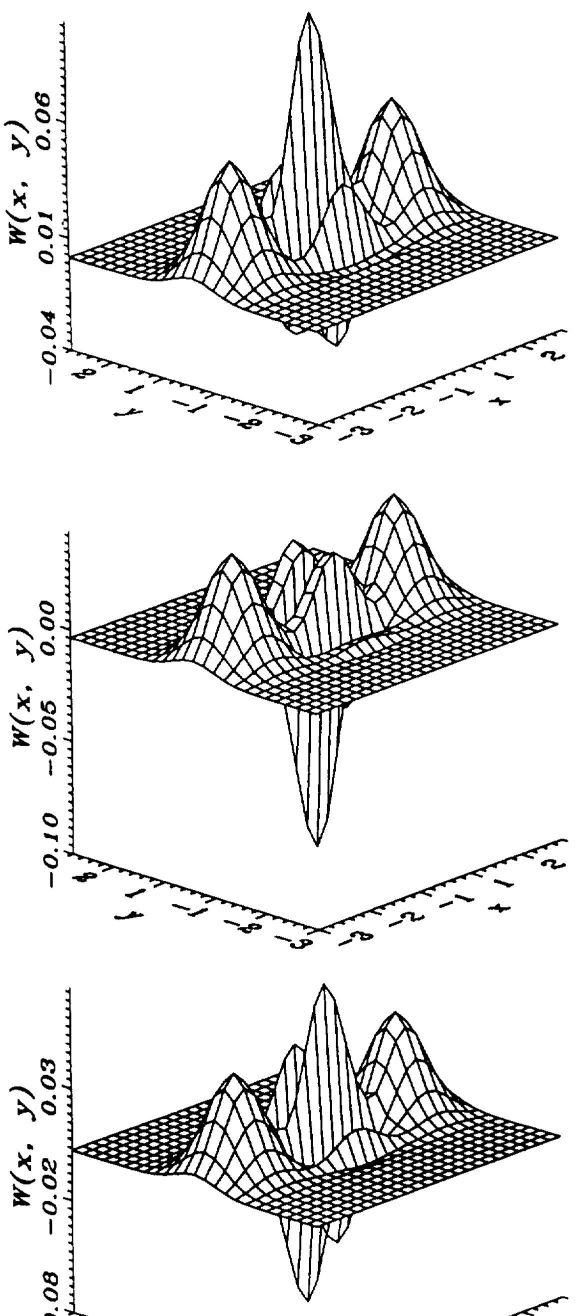

Figure 8: Wigner function corresponding to the superposition (44) of two coherent states with the relative phase ϕ equal to 0 (a); π (b) and $\pi/2$ (c); $\alpha = 2$.

where

$$W_{cl}(\beta) = \frac{4A}{\pi}\left\{e^{-2(x-\alpha)^2-2y^2} + e^{-2(x+\alpha)^2-2y^2}\right\} \tag{46}$$

and

$$W_{quant}(\beta) = \frac{4A}{\pi}\cos(4\alpha y + \phi)e^{-2x^2-2y^2}. \tag{47}$$

As earlier we use the notation $x = \mathrm{Re}\beta$; $y = \mathrm{Im}\beta$.

To investigate the dependence of the quantum interference on the value of the parameter ϕ we will employ two parameters describing nonclassical properties of light fields. Namely, we will study the Mandel Q parameter, defined as

$$Q = \frac{\langle(\Delta\hat{n})^2\rangle - \langle\hat{n}\rangle}{\langle\hat{n}\rangle} \tag{48}$$

which is related to the degree of sub-Poissonian photon statistics. In particular, if $Q = 0$ the state has Poissonian photon statistics, while for $Q < 0$ ($Q > 0$) the state has sub-Poissonian (super-Poissonian) photon statistics. The second parameter we will study is the squeezing parameter

$$S_i = 4\langle(\Delta\hat{a}_i)^2\rangle - 1 \tag{49}$$

describing the degree of quadrature squeezing. A state is said to be squeezed if S_1 or S_2 is less than zero. In what follows we will suppose three values of ϕ.

1) Let the phase ϕ be equal to zero. In this case the state (44) is equal to the even coherent state (8) and we find

$$Q = \frac{4\alpha^2\exp(-2\alpha^2)}{1-\exp(-4\alpha^2)} > 0; \tag{50}$$

$$S_1 = \frac{4\alpha^2}{1+\exp(-2\alpha^2)} > 0; \tag{51}$$

$$S_2 = -\frac{4\alpha^2\exp(-2\alpha^2)}{1+\exp(-2\alpha^2)} < 0, \tag{52}$$

from which it follows that the even coherent state has super-Poissonian photon statistics and simultaneously is squeezed in the $\hat{a}_2$ quadrature.

2) If $\phi = \pi$ then the state (44) is an odd coherent state [14]. This state has sub-Poissonian photon statistics, i.e.

$$Q = -\frac{4\alpha^2 \exp(-2\alpha^2)}{1 - \exp(-4\alpha^2)} < 0; \tag{53}$$

but is not squeezed

$$S_1 = \frac{4\alpha^2}{1 - \exp(-2\alpha^2)} > 0; \tag{54}$$

$$S_2 = -\frac{4\alpha^2 \exp(-2\alpha^2)}{1 - \exp(-2\alpha^2)} > 0. \tag{55}$$

3) Finally, if $\phi = 0$, then the state (44) has Poissonian photon statistics

$$Q = 0, \tag{56}$$

and simultaneously we can observe squeezing in the $\hat{a}_2$ quadrature

$$S_1 = 4\alpha^2; \tag{57}$$

$$S_2 = -4\alpha^2 \exp(-4\alpha^2) < 0. \tag{58}$$

The dependence of the statistical properties of superpositions of coherent states on the value of the relative phase is caused by the character of the quantum interference, that is whether this interference is constructive or destructive in various regions of phase space. This can be clearly seen from Figure 8 in which the Wigner function corresponding to the state (44) is plotted for $\phi = 0; \pi$ and $\pi/2$. We see significant differences in the shape of Wigner functions for various values of ϕ, which is related to the completely different statistical properties of the corresponding states.

5 Conclusion

The main information carried in this Lecture is: **A superposition of the most classical of field states can through the action of quantum interference, generate the archetypal nonclassical field states: the squeezed vacuum and the squeezed coherent state.**

Acknowledgements

This work was supported in part by the U.K. Science and Engineering Research Council. One of us (VB) would like to thank professor Y.S.Kim for hospitality at the University of Maryland.

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